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# The $N = 2$ supersymmetric heavenly equation and its super-hydrodynamical reduction

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## Abstract

Manifest  $N = 2$  supersymmetric Toda systems are constructed from the  $sl(n|n+1)$  superalgebras by taking into account their complex structure. In the  $n \rightarrow \infty$  continuum limit an  $N = 2$  extension of the  $(2+1)$ -dimensional heavenly equation is obtained. The integrability is guaranteed by the existence of a supersymmetric Lax pair. We further analyse the properties of the  $(1+1)$ -dimensionally reduced system. Its bosonic sector is of hydrodynamical type. This is not the case for the whole supersymmetric system which, however, is super-hydrodynamical when properly expressed in terms of a supergeometry involving superfields and fermionic derivatives.

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## 1. Introduction

The so-called ‘heavenly equation’ in  $(1+2)$  dimensions was first introduced to describe solutions of the complex Einstein equations [1]. It has received a lot of attention [2] for its remarkable integrability properties. Even if, to our knowledge, no attempt has been made so far in the literature to analyse the corresponding situation in the case of supergravity, the supersymmetric heavenly equation has been introduced and discussed in the context of superintegrable models. It appears [3] as the continuum limit of a system of equations known as the ‘supersymmetric Toda lattice hierarchy’, introduced in [4]. Such a system of equations and their hidden  $N = 2$  supersymmetric structures have been heavily investigated [5]. On the other hand, it has been recently pointed out that the dimensional reduction of the  $(1+2)$ -dimensional heavenly equation to a  $(1+1)$ -dimensional system is related to Benney-like integrable hierarchies of hydrodynamical equations [6]. Another very recent reference discussing the hydrodynamical properties of the reduced heavenly equation is [7].

In this paper we introduce the  $N = 2$  supersymmetric heavenly equation, clarifying its integrability properties and its algebra arising from the  $n \rightarrow \infty$  limit of a class of  $N = 2$  supersymmetric Toda equations. Further, we investigate its  $(1 + 1)$ -dimensional reduction which provides the supersymmetric hydrodynamical equations extending the Benney-like hierarchy of [6]. It allows us to show that the integrability properties of such systems are induced by the integrability property of the  $N = 2$  superheavenly equation and its Lax representation discussed below. This Lax representation is not in the form of a supersymmetric dispersionless Lax operator, as we discuss at length later.

In this paper we are not explicitly discussing the construction of the solutions of the (full and/or reduced) superheavenly equation from the given Lax pairs. This construction, however, can be straightforwardly carried out with the tools already available in the literature. As an example, explicit reconstruction formulae are furnished in [8] for the general class of (bosonic) Toda field theories. They can be extended to the  $N = 2$  supersymmetric systems as introduced in [9] and therefore applied to the  $N = 2$  Lax pair of the superheavenly equation given below in formula (12). Similar constructions are also available for the other Lax presentations discussed here.

The plan of the paper is as follows, in the next section we introduce the discretized  $N = 2$  Toda lattice hierarchy (as well as its continuum limit, the  $N = 2$  superheavenly equation), as a supersymmetric Toda system based on the  $sl(n|n + 1)$  superalgebra. It is worth noting that these are the same superalgebras originally employed in the construction of [3]. On the other hand, the superalgebras of the  $sl(n|n + 1)$  series admit a complex structure, allowing the construction of super-Toda systems based on  $N = 2$  superfields, according to the scheme of [9]. For this reason, the next section results provide a generalization of those of [3]. The supersymmetric Lax pairs providing integrability of the systems (for any given  $n$  and in the limit  $n \rightarrow \infty$ ) are explicitly constructed. They provide a dynamical formulation in the  $x_{\pm}$  plane (without involving the extra-time direction  $\tau$  of the superheavenly equation). In the following, we investigate the dimensional reduction of both the discretized and continuum systems from  $(1 + 2)$  to  $(1 + 1)$  dimensions. We obtain a supersymmetric system of equations with an interesting and subtle property. Unlike its purely bosonic subsector, the whole system involving fermions is not of hydrodynamical type. However, the same system, once expressed in terms of a supergeometry involving superfields and fermionic derivatives, satisfies a graded extension of the hydrodynamical property, which can be naturally named ‘super-hydrodynamical type of equation’. With this expression we mean that there is a way to recast the given  $(1 + 1)$ -dimensional supersymmetric equations into a system of non-linear equations for superfields which involves only first-order derivations w.r.t. the supersymmetric fermionic derivatives. This super-hydrodynamical system furnishes the supersymmetrization of the system introduced in [6].

## 2. The $N = 2$ superheavenly equation

The construction of a continuum limit (for  $n \rightarrow \infty$ ) of a discretized super-Toda system requires a presentation of the system in terms of a specific Cartan matrix. In the case of the series of the  $sl(n|n + 1)$  superalgebras, both a symmetric and an antisymmetric presentation for their Cartan matrices are available [10]. For our purposes, it turns out that the antisymmetric presentation is the one which works. The Cartan matrix  $a_{ij}$  of  $sl(n|n + 1)$  will therefore be chosen to be antisymmetric with the only non-vanishing entries given by  $a_{ij} = \delta_{i,i+1} - \delta_{i,i-1}$ .

The Cartan generators  $H_i$  and the fermionic simple roots  $F_{\pm i}$  satisfy

$$[H_i, F_{\pm j}] = \pm a_{ij} F_{\pm j} \quad \{F_i, F_{-j}\} = \delta_{ij} H_j. \quad (1)$$

The continuum limit of the construction [3] could have been performed for any superalgebra admitting an  $n \rightarrow \infty$  limit, such as  $sl(n|n)$ , etc. On the other hand, the superalgebras of the series  $sl(n|n+1)$  are special because they admit a complex structure and therefore the possibility of defining an  $N = 2$  manifestly supersymmetric Toda system, following the prescription of [9]. This is the content of the present section.

First we introduce the  $N = 2$  fermionic derivatives  $D_{\pm}, \bar{D}_{\pm}$ , acting on the  $x_{\pm} 2D$  spacetime ( $\theta_{\pm}$  and  $\bar{\theta}_{\pm}$  are Grassmann coordinates). The  $2D$  spacetime can be either Euclidean ( $x_{\pm} = x \pm t$ ) or Minkowskian ( $x_{\pm} = x \pm it$ ).

We have

$$D_{\pm} = \frac{\partial}{\partial \theta_{\pm}} - i\bar{\theta}_{\pm} \partial_{\pm} \quad \bar{D}_{\pm} = -\frac{\partial}{\partial \bar{\theta}_{\pm}} + i\theta_{\pm} \partial_{\pm}. \tag{2}$$

They satisfy the anticommutator algebra

$$\{D_{\pm}, \bar{D}_{\pm}\} = 2i\partial_{\pm} \tag{3}$$

and are vanishing otherwise.

Chiral ( $\Phi$ ) and antichiral ( $\bar{\Phi}$ )  $N = 2$  superfields are respectively constrained to fulfil the conditions

$$\bar{D}_{\pm} \Phi = 0 \quad D_{\pm} \bar{\Phi} = 0. \tag{4}$$

Accordingly, a generic chiral superfield  $\Phi$  is expanded in its bosonic  $\varphi, F$  (the latter is auxiliary) and fermionic component fields ( $\psi_+, \psi_-$ ) as

$$\Phi(\hat{x}_{\pm}, \theta_{\pm}) = \varphi + \theta_+ \psi_+ + \theta_- \psi_- + \theta_+ \theta_- F \tag{5}$$

with  $\varphi, \psi_{\pm}$  and  $F$  evaluated in  $\hat{x}_{\pm} = x_{\pm} + i\bar{\theta}_{\pm} \theta_{\pm}$ .

Similarly, the antichiral superfield  $\bar{\Phi}$  is expanded as

$$\bar{\Phi}(\bar{x}_{\pm}, \bar{\theta}_{\pm}) = \bar{\varphi} + \bar{\theta}_+ \bar{\psi}_+ + \bar{\theta}_- \bar{\psi}_- + \bar{\theta}_+ \bar{\theta}_- \bar{F} \tag{6}$$

with all component fields evaluated in  $\bar{x}_{\pm} = x_{\pm} - i\bar{\theta}_{\pm} \theta_{\pm}$ .

Due to the complex structure of  $sl(n|n+1)$ , its Cartan and its simple (positive and negative) root sector can be split into its conjugated parts

$$\begin{aligned} \mathcal{H} &\equiv \{H_{2k-1}\} & \bar{\mathcal{H}} &\equiv \{H_{2k}\} \\ \mathcal{F}_+ &\equiv \{F_{2k-1}\} & \mathcal{F}_- &\equiv \{F_{-(2k-1)}\} \\ \bar{\mathcal{F}}_+ &\equiv \{F_{-2k}\} & \bar{\mathcal{F}}_- &\equiv \{F_{2k}\} \end{aligned} \tag{7}$$

for  $k = 1, 2, \dots, n$ .

Following [9], we can introduce the  $sl(n|n+1) N = 2$  super-Toda dynamics, defined for the Cartan-valued chiral ( $\Phi$ ) and antichiral ( $\bar{\Phi}$ )  $N = 2$  superfields,

$$\Phi = \sum_{k=1}^n \Phi_k H_{2k-1} \quad \bar{\Phi} = \sum_{k=1}^n \bar{\Phi}_k H_{2k} \tag{8}$$

through the Lax operators  $\mathcal{L}_{\pm}$  and  $\bar{\mathcal{L}}_{\pm}$ , given by

$$\mathcal{L}_+ = D_+ \Phi + e^{\bar{\Phi}} F_+ e^{-\bar{\Phi}} \quad \mathcal{L}_- = -F_- \tag{9}$$

and

$$\bar{\mathcal{L}}_+ = \bar{F}_+ \quad \bar{\mathcal{L}}_- = \bar{D}_- \bar{\Phi} + e^{\Phi} \bar{F}_- e^{-\Phi} \tag{10}$$

where

$$\begin{aligned} F_+ &= \sum_k F_{2k-1} & F_- &= \sum_k F_{-(2k-1)} \\ \bar{F}_+ &= \sum_k F_{-(2k)} & \bar{F}_- &= \sum_k F_{2k} \end{aligned} \tag{11}$$

(as before, the sum is over the positive integers up to  $n$ ).

Explicitly, we have

$$\begin{aligned}
 \mathcal{L}_+ &= \sum_k (D_+ \Phi_k H_{2k-1} + e^{\bar{\Phi}_{k-1} - \bar{\Phi}_k} F_{2k-1}) \\
 \mathcal{L}_- &= - \sum_k F_{-(2k-1)} \\
 \bar{\mathcal{L}}_+ &= \sum_k F_{-2k} \\
 \bar{\mathcal{L}}_- &= \sum_k (\bar{D}_- \bar{\Phi}_k H_{2k} + e^{\Phi_k - \Phi_{k+1}} F_{2k}).
 \end{aligned}
 \tag{12}$$

Please note that, in order to have a more compact notation, here and in the following we have formally introduced the superfields  $\bar{\Phi}_0$  and  $\Phi_{n+1}$ , both set equal to zero ( $\bar{\Phi}_0 \equiv 0 \equiv \Phi_{n+1}$ ).

The zero-curvature equations are given by

$$\begin{aligned}
 D_+ \mathcal{L}_- + D_- \mathcal{L}_+ + \{\mathcal{L}_+, \mathcal{L}_-\} &= 0 \\
 \bar{D}_+ \bar{\mathcal{L}}_- + \bar{D}_- \bar{\mathcal{L}}_+ + \{\bar{\mathcal{L}}_+, \bar{\mathcal{L}}_-\} &= 0
 \end{aligned}
 \tag{13}$$

so that the following set of equations for the constrained (anti)chiral  $N = 2$  superfields is obtained:

$$D_+ D_- \Phi_k = -e^{\bar{\Phi}_{k-1} - \bar{\Phi}_k} \quad \bar{D}_+ \bar{D}_- \bar{\Phi}_k = -e^{\Phi_k - \Phi_{k+1}}
 \tag{14}$$

for the positive integers  $k = 1, 2, \dots, n$ .

By setting

$$B_k = \Phi_k - \Phi_{k+1} \quad \bar{B}_k = \bar{\Phi}_k - \bar{\Phi}_{k+1}
 \tag{15}$$

we get the systems of equations

$$D_+ D_- B_k = e^{\bar{B}_k} - e^{\bar{B}_{k-1}} \quad \bar{D}_+ \bar{D}_- \bar{B}_k = e^{B_{k+1}} - e^{B_k}
 \tag{16}$$

for  $k$  restricted to  $k = 1, 2, \dots, n - 1$ , together with

$$D_+ D_- B_n = -e^{\bar{B}_{n-1}} \quad \bar{D}_+ \bar{D}_- \bar{B}_0 = e^{B_1}.
 \tag{17}$$

By identifying  $k$  as a discretized extra time-like variable  $\tau$  we obtain, in the continuum limit for  $n \rightarrow \infty$

$$D_+ D_- B = \partial_\tau e^{\bar{B}} \quad \bar{D}_+ \bar{D}_- \bar{B} = \partial_\tau e^B
 \tag{18}$$

which corresponds to the  $N = 2$  extension of the superheavenly equation.

Indeed, the presence in the previous equations of the first derivative in  $\tau$  is an artifact of the  $N = 2$  superfield formalism. Once the equations at the level of the component fields are solved and the auxiliary fields eliminated in terms of the equations of motion, we are left with a system of second-order equations.

We have, in terms of the component fields,

$$\begin{aligned}
 B &= (1 + i\bar{\theta}_+ \theta_+ \partial_+ + i\bar{\theta}_- \theta_- \partial_- - \bar{\theta}_+ \theta_+ \bar{\theta}_- \theta_- \partial_+ \partial_-) C \\
 \bar{B} &= (1 - i\bar{\theta}_+ \theta_+ \partial_+ - i\bar{\theta}_- \theta_- \partial_- - \bar{\theta}_+ \theta_+ \bar{\theta}_- \theta_- \partial_+ \partial_-) \bar{C}
 \end{aligned}
 \tag{19}$$

where

$$\begin{aligned}
 C &= (b + \theta_+ \psi_+ + \theta_- \psi_- + \theta_+ \theta_- a) \\
 \bar{C} &= (\bar{b} + \bar{\theta}_+ \bar{\psi}_+ + \bar{\theta}_- \bar{\psi}_- + \bar{\theta}_+ \bar{\theta}_- \bar{a})
 \end{aligned}
 \tag{20}$$

with  $a, \bar{a}$  bosonic auxiliary fields. All component fields are evaluated in  $x_\pm$  only.

The equations of motion of the  $N = 2$  superheavenly equation are explicitly given in components by

$$\begin{aligned}
 -a &= (e^{\bar{b}})_\tau \\
 2i\partial_- \psi_+ &= (\bar{\psi}_- e^{\bar{b}})_\tau \\
 -2i\partial_+ \psi_- &= (\bar{\psi}_+ e^{\bar{b}})_\tau \\
 -4\partial_+ \partial_- b &= (\bar{a} e^{\bar{b}} - \bar{\psi}_+ \bar{\psi}_- e^{\bar{b}})_\tau
 \end{aligned} \tag{21}$$

and

$$\begin{aligned}
 -\bar{a} &= (e^b)_\tau \\
 2i\partial_- \bar{\psi}_+ &= (\psi_- e^b)_\tau \\
 -2i\partial_+ \bar{\psi}_- &= (\psi_+ e^b)_\tau \\
 -4\partial_+ \partial_- \bar{b} &= (a e^b - \psi_+ \psi_- e^b)_\tau.
 \end{aligned} \tag{22}$$

Eliminating the auxiliary fields we obtain the systems

$$\begin{aligned}
 2i\partial_- \psi_+ &= (\bar{\psi}_- e^{\bar{b}})_\tau \\
 -2i\partial_+ \psi_- &= (\bar{\psi}_+ e^{\bar{b}})_\tau \\
 4\partial_+ \partial_- b &= ((e^{\bar{b}})_\tau e^{\bar{b}} + \bar{\psi}_+ \bar{\psi}_- e^{\bar{b}})_\tau
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 2i\partial_- \bar{\psi}_+ &= (\psi_- e^b)_\tau \\
 -2i\partial_+ \bar{\psi}_- &= (\psi_+ e^b)_\tau \\
 4\partial_+ \partial_- \bar{b} &= ((e^{\bar{b}})_\tau e^b + \psi_+ \psi_- e^b)_\tau.
 \end{aligned} \tag{24}$$

The bosonic component fields  $b, \bar{b}$ , as well as the fermionic ones  $\psi_\pm, \bar{\psi}_\pm$ , are all independent. The equations (23), (24) are a manifestly  $N = 2$  supersymmetric extension of the system introduced in [3].

### 3. Super-hydrodynamical reductions of the superheavenly equation

The equations (23) and (24) correspond to a  $(1 + 2)$ -dimensional system, manifestly relativistic and  $N = 2$  supersymmetric w.r.t. the two-dimensional subspace spanned by the  $x_\pm$  coordinates, while possessing an extra bosonic time-like dimension denoted as  $\tau$ . A very interesting example of integrable system in  $(1 + 1)$  dimension which is currently intensely investigated, see e.g. [6, 7], can be recovered by applying a dimensional reduction to, let us say, the bosonic sector of the  $(1 + 2)$ -dimensional heavenly equation. We can refer to such a system, perhaps a bit pedantically, as the  $(1 + 1)$ -dimensionally reduced heavenly equation. It can be obtained by setting equal to zero the fermionic variables  $\psi_\pm, \bar{\psi}_\pm \equiv 0$ , while  $b, \bar{b}$ , can be consistently constrained as  $\bar{b} = b$ . The  $x_\pm$  coordinates are identified, i.e.  $x_+ = x_- = x$ .

The resulting equation, by changing the normalizations (setting  $f = 2b$  and  $t = 2\tau$ ) can be conveniently written as

$$f_{tt} = (e^f)_{xx}. \tag{25}$$

This equation has recently received a lot of attention in the literature. It is an example of a completely integrable, hydrodynamical type equation. It admits a multi-Hamiltonian structure

and possesses an infinite number of conserved charges in involution. It has been recently shown that it can be recovered via a dispersionless Lax representation given by

$$L := p^{-1}(1 + gp^2 + hp^4)^{\frac{1}{2}} \tag{26}$$

with  $f, g$  functions of  $x, t$ , while  $p$  is the classical momentum. The equations of motion are read from

$$\frac{\partial L}{\partial t} = \frac{p}{2} \{L_{\leq 0}^2, L\} \tag{27}$$

where  $\{*, *\}$  denotes the usual Poisson brackets and  $L_{\leq 0}^2$ , defined in [6], is explicitly given by  $L_{\leq 0}^2 = p^{-2} + g$ .

Equation (27) leads to

$$\frac{\partial h}{\partial t} = hg_x \quad \frac{\partial g}{\partial t} = h_x \tag{28}$$

Equation (25) is recovered after eliminating  $g$  from the previous system and setting  $h = e^f$ . In (28) form, the ‘(1 + 1)-dimensionally reduced heavenly equation’ is shown to be a hydrodynamical type of equation.

An interesting question, whose solution as we will see is non-trivial, is whether the (1 + 1)-dimensional reduction (for  $x_+ = x_- = x$ ) of the full  $N = 2$  supersymmetric heavenly equation is also of hydrodynamical type. The answer is subtle. The introduction of the fermionic fields  $\psi_{\pm}, \bar{\psi}_{\pm}$ , whose equations of motion are of first order in the extra-bosonic time, does not allow the dimensionally reduced system from (23), (24) to be represented in hydrodynamical form, due to a mismatch with the second-order equation satisfied by the bosonic fields  $b, \bar{b}$ . On the other hand, it is quite natural to expect that important properties are not spoiled by the supersymmetrization. This is indeed the case with the hydrodynamical reduction, when properly understood. The key issue here is the fact that the nice features of the supersymmetry are grasped when inserted in the proper context of a supergeometry, which must be expressed through the use of a superfield formalism. It is in this framework that a super-hydrodynamical reduction of the dimensionally reduced system from (23), (24) becomes possible. Indeed, for  $x_+ = x_- = x$ , we can write down the (18) system through the set of, first order in the fermionic derivatives, *superfield* equations

$$D_- B = N_{\tau} \quad D_+ N = e^{\bar{B}} \tag{29}$$

and

$$\bar{D}_- \bar{B} = \bar{N}_{\tau} \quad \bar{D}_+ \bar{N} = e^B \tag{30}$$

with  $N$  and  $\bar{N}$  subsidiary fermionic superfields. Taking into account the following expansions:

$$\begin{aligned} e^{\bar{B}} &= (1 - i\bar{\theta}_+ \theta_+ \partial_+ - i\bar{\theta}_- \theta_- \partial_- - \bar{\theta}_+ \theta_+ \bar{\theta}_- \theta_- \partial_+ \partial_-) e^{\bar{C}} \\ &= D_+ \theta_+ (1 - i\bar{\theta}_- \theta_- \partial_-) e^{\bar{C}} \\ e^B &= (1 + i\bar{\theta}_+ \theta_+ \partial_+ + i\bar{\theta}_- \theta_- \partial_- - \bar{\theta}_+ \theta_+ \bar{\theta}_- \theta_- \partial_+ \partial_-) e^C \\ &= -\bar{D}_+ \bar{\theta}_+ (1 + i\bar{\theta}_- \theta_- \partial_-) e^C \end{aligned} \tag{31}$$

we easily obtain the following solutions for  $N, \bar{N}$

$$\begin{aligned} N &= (\theta_+ - i\theta_+ \bar{\theta}_- \theta_- \partial_-) e^{\bar{C}} + D_+ \Omega_1 \\ \bar{N} &= -(\bar{\theta}_+ + i\bar{\theta}_+ \bar{\theta}_- \theta_- \partial_-) e^C + \bar{D}_+ \bar{\Omega}_1 \end{aligned} \tag{32}$$

in terms of arbitrary  $\Omega_1, \bar{\Omega}_2$  bosonic superfunctions.

If we set, as we are free to choose,

$$\begin{aligned}\Omega_1 &= \theta_+(\psi_- - 2i\bar{\theta}_- b_- - i\bar{\theta}_- \theta_- \partial_- \psi_-) \\ \bar{\Omega}_1 &= \bar{\theta}_+(\bar{\psi}_- - 2i\theta_- \bar{b}_- + i\bar{\theta}_- \theta_- \partial_- \bar{\psi}_-)\end{aligned}\tag{33}$$

and substitute the values of  $N, \bar{N}$  given in (32) back to (29) and (30), we obtain the equivalence of this system of equations w.r.t. the  $x_+ = x_- = x$  dimensional reduction of equations (23) and (24). This proves the existence of a super-hydrodynamical reduction expressed in a superfield formalism. If we express this set of equations in terms of the component fields, as mentioned before, we are not obtaining a hydrodynamical type equation. This fact should not be regarded as a vicious feature of our system, but rather as a virtue of the supersymmetry. In several examples, this is one, the introduction of the super-formalism allows both the properties of the systems, and the techniques used to investigate them, to be extended to cases for which the ordinary methods are of no help. In a somehow related area we can cite, e.g., the derivation of bosonic integrable hierarchies associated with the bosonic sector of super-Lie algebras [11].

Let us finally discuss the restriction from  $N = 2 \rightarrow N = 1$ , i.e. down to the  $N = 1$  supersymmetry. It is recovered from setting

$$\begin{aligned}\bar{\theta}_+ &= -\theta_+ & \bar{\theta}_- &= -\theta_- \\ \bar{D}_+ &= -D_+ & \bar{D}_- &= -D_-\end{aligned}\tag{34}$$

Equations (23) and (24) now read

$$D_+ D_- C = \partial_\tau e^{\bar{C}} \quad D_+ D_- \bar{C} = \partial_\tau e^C\tag{35}$$

where now  $D_\pm$  denote the  $N = 1$  fermionic derivatives

$$D_\pm = \frac{\partial}{\partial \theta_\pm} + i\theta_\pm \partial_\pm\tag{36}$$

Equations (35) have a similar component structure to equations (23) and (24). We have indeed

$$\begin{aligned}\partial_- \psi_+ &= i(\bar{\psi}_- e^{\bar{b}})_\tau \\ \partial_+ \psi_- &= -i(\bar{\psi}_+ e^{\bar{b}})_\tau \\ \partial_+ \partial_- b &= ((e^{\bar{b}})_\tau e^{\bar{b}} + \bar{\psi}_+ \bar{\psi}_- e^{\bar{b}})_\tau\end{aligned}\tag{37}$$

and

$$\begin{aligned}\partial_- \bar{\psi}_+ &= i(\psi_- e^b)_\tau \\ \partial_+ \bar{\psi}_- &= -i(\psi_+ e^b)_\tau \\ \partial_+ \partial_- \bar{b} &= ((e^{\bar{b}})_\tau e^b + \psi_+ \psi_- e^b)_\tau.\end{aligned}\tag{38}$$

If we further reduce our system by constraining the variables  $x_\pm$  ( $x_+ = -x_- = it$ ) and setting  $\bar{C} = C$ , we obtain the following system

$$\begin{aligned}\partial_t \psi_+ &= \partial_\tau (e^b \psi_-) \\ \partial_t \psi_- &= \partial_\tau (e^b \psi_+) \\ \partial_t^2 b &= \partial_\tau (\frac{1}{2}(e^{2b})_\tau + e^b \psi_+ \psi_-).\end{aligned}\tag{39}$$

As its  $N = 2$  counterpart, this system is not of hydrodynamical type. However, it can be easily shown, with a straightforward modification of the procedure previously discussed for the  $N = 2$  case, to be super-hydrodynamical when expressed in terms of superfields and superderivatives.



#### 4. Conclusions

In this paper we have first introduced manifest  $N = 2$  supersymmetric discrete Toda equations in association with the complex-structure superalgebras of the  $sl(n|n + 1)$  series. We proved that in the  $n \rightarrow \infty$  limit the discrete variable can be regarded as a continuum extra bosonic time  $\tau$ . The corresponding system is a manifestly  $N = 2$  supersymmetric extension of the standard heavenly equation in  $(2 + 1)$  dimensions, the supersymmetry being associated with the plan spanned by the bosonic  $x_{\pm}$  coordinates and their fermionic counterparts, which are naturally expressed in an  $N = 2$  superspace formalism. A supersymmetric Lax-pair, proving the integrability of the system, was constructed. It expresses the dynamics w.r.t. the  $\pm$  superspace coordinates ( $\tau$  in this respect can be considered either as an auxiliary parameter or as the discrete label of the original formulation).

We further investigated the dimensional reduction of the above systems from the  $(2 + 1)$ -dimensional to the  $(1 + 1)$ -dimensional case. In the purely bosonic case the reduced systems of equations are of a special type, they are hydrodynamical systems of non-linear equations. The supersymmetric case is much subtler. This system is not of hydrodynamical type since the presence of the fermions spoils this property. However, when properly expressed in terms of a supergeometry (essentially, superfields and fermionic derivatives) it is of superhydrodynamical type (the precise meaning of this term has been explained in the introduction). More than a vice, this can be considered as a virtue of the supersymmetry. It allows the notion of hydrodynamical equations to be extended beyond the realm of the systems ordinarily allowed.

In the present paper we did not discuss any properties of the solutions of our system. However, as recalled in the introduction, available tools exist in the literature (see e.g. [8]) for explicitly constructing solutions from given Lax pairs of the same type as ones considered here. The analysis of this procedure, rather straightforward, will be left for future works.

Finally, it is worth mentioning a formal, however challenging problem, concerning the supersymmetrization. While the integrability of the systems under consideration is automatically guaranteed by the given supersymmetric and relativistic Lax-pairs in the  $\pm$  plane mentioned above, one can wonder whether a dispersionless Lax operator, directly expressing the  $\tau$  dynamics (i.e., in terms of the bosonic extra-time) could be found. The answer is indeed positive for the purely bosonic sector. In the supersymmetric case, however, no closed dispersionless Lax operator is known at present. It is repeated here that an analogous situation has already been encountered for the polytropic gas systems [12]. The problem of constructing supersymmetric dispersionless Lax operators for these related systems is still open.

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